Finite Element Methods with Higher Order Polynomials

Konstantina C. Kyriakoudi and Michail A. Xenos

Abstract The Finite Element Method (FEM) has recently been implemented in the fluid mechanics field to solve the instabilities that arise as a result of the equations' non-linearities. For this reason, novel formulations of FEM were introduced, including the use of orthogonal polynomials and high-order polynomials. In this review, the focus rests on studying and analysing the aforementioned formulations and describing their improvements over the classical method. Initially, a theoretical background of FEM is introduced, with an emphasis on evaluating the basis of the function space. Additionally, the *p-version* of FEM is analysed, using Legendre polynomials. A comparison of the classical *h-version* and the *p-version* in terms of convergence. Moreover, other formulations that yield, using higher-order polynomials, such as *hp-FEM* and Spectral Element Method, are briefly reviewed. Finally, applications on FEM are presented, revealing the effects of the increase in the degree of the polynomials when solving a fluid mechanics problem.

Key words: Finite Element Method, p-version, adaptive, Fluid Mechanics

AMS Subject Classification: 65N30, 65M60, 76M10

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1 Introduction

Engineering problems sometimes can be complicated to solve analytical because the partial differential equations that describe them. Thus, numerical methods need to be applied to solve those differential equations. A widely known method for such problems is the finite element method (FEM). The FEM were initially applied mainly in the field of structural mechanics and later the approach was extended to include applications in fluid mechanics, where convective terms play a significant part and contribute to a non-linear formulation of the problem. The non-linearities and instabilities in the solutions to these issues slowed down the progress of the method in fluid mechanics.

In the finite element method application, the design of the mesh and the choice of the elements are two of the most important considerations. In the classical approach, piecewise polynomials of fixed degree p are utilized and the mesh size h is decreased for accuracy. This method is called the *h-version* of FEM. On the other hand, the *p-version* refers to a fixed mesh and p is allowed to increase and the *hp-version* combines both the approaches. [2].

The *h-version* of the finite element method commonly uses linear or quadratic Lagrangian-interpolant shape functions. The accuracy of this method can be improved while increasing the number of elements in the domain resulting in decreasing the element size. [20]. The *p-version* of the finite element method is a variation of the classical approach where the approximation is represented by higher-order polynomials. The *hp-version* or Spectral methods benefit from modifications in both the mesh and the polynomial degree.

The review starts with the classical formulation of the finite element method. Then the different variations of FEM are presented, beginning with the *p-version* and the *hp-version*. Additionally, convergence is briefly studied for the aforementioned variations. The various types of refinement are then described in the section on adaptive mesh refinement. Some popular fluid mechanics problems are presented in the applications part. Decisively the primary emphasis is on aiming at increasing the polynomial degree of the equations and observing the accuracy of the approach.

2 Finite Element Method

Ritz, a German scientist, formulated the fundamental concepts of the FEM in 1909 for approximating solutions for problems with flexible solid mechanics. His method included estimating an energy functional while using known functions multiplied by unidentified coefficients. A system finds those coefficients by the minimization of the functional in relation to each unknown. The functions that can satisfy the provided boundary conditions are constrained. Courant, in 1943, developed Ritz's method further by introducing linear functions defined over triangular regions. The term Finite Element was introduced in 1960 by Clough [18]. The main concept of

FEM is to replace any continuous function with an approximation in a discrete space where a set of polynomials is used to describe them.

2.1 Basic Theory of the *h-version*

Consider the following boundary value problem

$$\begin{cases} -\dfrac{d^2u}{dx^2} = f, x \in (0,1) \\ u(0) = 0 \\ u'(1) = 0 \end{cases} \tag{1}$$

The first step is to multiply both sides with a test function v and integrate. This yields to the **weak form** of the problem:

$$a(u,v) = (f,v), \quad \forall v \in V. \tag{2}$$

where

$$a(u,v) = \int_0^1 u'(x)v'(x)dx \tag{3}$$

The quantities u, v in the weak form can be scalar or vector functions depending the dimensions.

Let us define the function space

$$V = \left\{ v \in L^2(0,1) : a(v,v) < \infty \;\; and \;\; v(0) = 0 \right\}$$

as the **test space**, and the space of **square integrable** functions in $[0,1]$ is noted by $L^2(0,1)$.

It can be proved that these functions can create a Banach space. In the case that it is enforced with an inner product, then the space is called Hilbert (and is the same as a H^1 Sobolev space).

The Lax-Milgram theorem ensures existence and uniqueness of the solution for both the variational and the approximation problems [4, 5].

Theorem 1 ***(Lax-Milgram)***

Given a Hilbert space $(V, (\cdot,\cdot))$, a continuous, coercive bilinear form $a(\cdot,\cdot)$ and a continuous linear functional $F \in V'$, there exists a unique solution $u \in V$, such that,

$$a(u,v) = F(v), \quad \forall\, v \in V. \tag{4}$$

In higher dimensional problems, the variational form becomes

$$a(u, v) = \int_{\Omega} A(x)\nabla u(x) \cdot \nabla v(x) + (\mathbf{B}(x) \cdot \nabla u(x))\, v(x) + C(x)u(x)v(x)\, dx \quad (5)$$

where $A, \mathbf{B}, C$ are bounded and measurable functions on $\Omega \subset \mathbb{R}^n$ and $\mathbf{B}$ is a vector.

The uniqueness of the solution, in this case, is guaranteed [4, 18].

2.2 Shape Functions

Due to the utilization of Sobolev spaces, some functions are discontinuous. Therefore, there is a need to focus on Piecewise Polynomial Spaces. Let a partition of $[a, b]$ with n elements $0 = x_0 < x_1 < \ldots < x_n = 1$ and let V_h be a linear space of functions v such that:

- $v \in C^0([0, 1])$
- $v\,|_{[x_{i-1}, x_i]}$ is a linear polynomial and
- $v(0) = 0$

All ϕ_i functions can be defined, for all $i = 1, \ldots, n$ and $\phi_i(x_j) = \delta_{ij}$, Kronecker's delta. The purpose of this space is to construct an orthonormal basis $\{\phi_i : 1 \leq i \leq n\}$ for the V_h space. This is called **nodal basis** and the points x_i are called **nodes**.

The discrete space creates the necessity to describe the functions using a proper basis. The fact that the coordinates of each element can periodically make the task more complex demands finding a more effective technique to perform our calculations. Thus, the following index is provided to overcome those challenges, by transferring the global system to the local, in the interval $[0, 1]$:

$$i(e, j) = e + j - 1,$$

The discrete form of the functions involved in the problem (2) is presented in the following definitions.

Definition 1 Given $v \in C^0([0, 1])$, then $v_I \in V_h$ is the interpolant of v and is determined by

$$v_I := \sum_{i=1}^{n} v(x_i)\phi_i.$$

In the same fashion, the interpolant of f, f_I, is defined :

$$f_I := \sum_{e} \sum_{j=0}^{1} f\left(x_{i(e,j)}\right) \phi_j^e \quad (6)$$

where $\{\phi_j^e : j = 0, 1\}$ is the basis of the interval $I_e = [x_{e-1}, x_e]$:

$$\phi_j^e(x) = \phi_j((x - x_{e-1})/(x_e - x_{e-1}))$$

and

$$\phi_0(x) = \begin{cases} 1 - x, & x \in [0,1] \\ 0, & \text{else} \end{cases} \qquad \phi_1(x) = \begin{cases} x, & x \in [0,1] \\ 0, & \text{else} \end{cases}$$

At last the bilinear form $a(u, v)$ is converted to:

$$a(u, v) = \sum_e a_e(u, v)$$

where $a_e(u, v)$ is the local bilinear form in each element defined by the following

$$\begin{aligned} a_e(u, v) &:= \int_{I_e} u'v' \, dx \\ &= (x_e - x_{e-1})^{-1} \int_0^1 \left(\sum_j u_{i(e,j)}\phi_j\right)' \left(\sum_j v_{i(e,j)}\phi_j\right)' \, dx \\ &= (x_e - x_{e-1})^{-1} \begin{pmatrix} u_{i(e,0)} \\ u_{i(e,1)} \end{pmatrix}^t K \begin{pmatrix} v_{i(e,0)} \\ v_{i(e,1)} \end{pmatrix}, \end{aligned}$$

where K is the local stiffness matrix

$$K_{i,j} := \int_0^1 \phi'_{i-1}\phi'_{j-1} \, dx, \quad i, j = 1, 2.$$

The solution to the problem results from solving the above system.

2.3 *p-version* and Hierarchical Basis

The classical form of the finite element method utilizes low-order polynomials such as the Lagrange, to create the basis which is called the standard, whereas in the *p-version* the degree of polynomials can increase using orthogonal polynomials such as Legendre or Chebyshev polynomials. The resulting base is called hierarchical. The main difference with respect to the Lagrange polynomials, commonly used in low-order finite elements, consists of accounting for the lower-order basis functions when higher-order shape functions are computed [19, 20, 24].

The selection of shape functions is crucial in defining the finite element space. However, there are several factors to consider when choosing shape functions according to Szabó [22]. To begin with, the shape functions should have as small of an error as possible when mapped to increasing polynomial degrees. It is important for shape functions to allow efficient calculations of the stiffness matrices and permit minimal continuity error. Also, shape functions selection can greatly impact the implementation of iteration procedures, especially for larger problems. Therefore, polynomials with specific orthogonality properties should be preferred for the construction of the shape functions. These properties are to be hierarchic, this means that the set of polynomial degree $p + 1$ should contain the set of shape functions of polynomial degree p. With this is mind, the number of shape functions should be as small as possible, especially at vertices, edges, and faces.

The hierarchic shape functions are the integrated Legendre shape functions defined as:

$$
\begin{aligned}
N_1(\xi) &= \frac{1}{2}(1-\xi) \\
N_2(\xi) &= \frac{1}{2}(1+\xi) \\
N_i(\xi) &= \phi_{i-1}(\xi), \quad i = 3, 4, \ldots p+1
\end{aligned}
$$

where ϕ_{i-1} is computed using the expression:

$$
\begin{aligned}
\phi_j(\xi) &= \sqrt{\frac{2j-1}{2}} \int_{-1}^{\xi} L_{j-1}(x)\, dx \\
&= \frac{1}{\sqrt{4j-2}} \left(L_j(\xi) - L_{j-2}(\xi) \right) \quad j = 2, 3, \ldots
\end{aligned}
$$

where $L_j(\xi)$ are the Legendre polynomials. Since they are computed by means of the Legendre polynomials, they follow its orthogonality property:

$$
\int_{-1}^{1} \frac{dN_i}{d\xi} \frac{dN_j}{d\xi} d\xi = \delta_{ij} \quad i \geq 3 \text{ and } j \geq 1 \quad \text{or vice-versa} \tag{7}
$$

The first five degrees of hierarchical shape functions for one-dimensional problems are the following:

$$
\begin{aligned}
N_1(\xi) &= \frac{1}{2}(1-\xi) & N_4(\xi) &= \frac{1}{4}\sqrt{10}(\xi^2-1)\xi \\
N_2(\xi) &= \frac{1}{2}(1+\xi) & N_5(\xi) &= \frac{1}{16}\sqrt{14}(5\xi^2-6\xi+1) \\
N_3(\xi) &= \frac{1}{4}\sqrt{6}(\xi^2-1) & N_6(\xi) &= \frac{1}{16}\sqrt{2}(7\xi^2-10\xi+3)\xi
\end{aligned}
$$

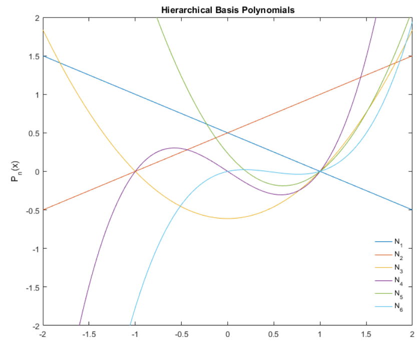


Fig. 1: Hierarchical Basis from Legendre polynomials

The first two shape functions N_1, N_2 are called *nodal shape functions* or *nodal modes*. The next shape functions $N_i(\xi), i = 3, 4, \ldots$ are called *internal shape functions*, *internal modes*, or *bubble modes*, because $N_i(-1) = N_i(1) = 0$. The main difference here is that all the shape functions of lower order are embedded in the hierarchical basis.

Thus, the functions can be written similarly to before but with the hierarchical basis functions. The bilinear form is computed from

$$\begin{aligned} a_1(u, v) &= \frac{2}{x_{e+1} - x_e} \int_{-1}^{+1} \kappa(Q_e(\xi)) \left(\sum_{j=1}^{p_e+1} a_j \frac{dN_j}{d\xi} \right) \left(\sum_{i=1}^{p_e+1} b_i \frac{dN_i}{d\xi} \right) d\xi \\ &= \sum_{i=1}^{p_e+1} \sum_{j=1}^{p_e+1} k_{ij}^e a_j b_i \\ &= b^T K^e a \end{aligned}$$

where

$$k_{ij}^e = \frac{2}{x_{e+1} - x_e} \int_{-1}^{+1} \kappa(Q_e(\xi)) \frac{dN_i}{d\xi} \frac{dN_j}{d\xi} d\xi$$

The stiffness matrix K is symmetric following the symmetry of the bilinear form and the fact that the same basis is used for u, v. Following the same procedure the second term of the bilinear form is given from:

$$
\begin{aligned}
a_2(u,v) &= \frac{x_{e+1}-x_e}{2}\int_{-1}^{+1} c(Q_e(\xi))\left(\sum_{j=1}^{p_e+1} a_j N_j\right)\left(\sum_{i=1}^{p_e+1} b_i N_i\right) d\xi \\
&= \sum_{i=1}^{p_e+1}\sum_{j=1}^{p_e+1} m_{ij}^e a_j b_i \\
&= b^T M^e a
\end{aligned}
$$

where

$$
m_{ij}^e = \frac{x_{e+1}-x_e}{2}\int_{-1}^{+1} c(Q_e(\xi)) N_i N_j d\xi
$$

When using hierarchic shape functions, the stiffness matrix tends to become almost perfectly diagonal. [8, 21]

The right-hand side of the equation involves the numerical evaluation of $f(v)$

$$
\begin{aligned}
f(v) &= \frac{x_{e+1}-x_e}{2}\int_{-1}^{+1} f(Q_e(\xi))\left(\sum_{i=1}^{p_e+1} b_i^e N_i\right) d\xi \\
&= \sum_{i=1}^{p_e+1} b_i^e r_i^e
\end{aligned}
$$

where

$$
r_i^e = \frac{x_{e+1}-x_e}{2}\int_{-1}^{+1} f(Q_e(\xi)) N_i(\xi) d\xi
$$

In the two-dimensional case, three kinds of shape functions are involved. These are *nodal*, *edge*, and *internal* shape functions. Accordingly, elements in three dimensions are characterized by one more group of shape functions, the *face nodes*. [1, 21, 22]

2.4 Spectral and *hp* Methods

There is also a formulation where the mesh and the degree of the polynomials can change [12]. Spectral methods, which were introduced by Patera [17], combine both of the previous versions of the Finite Element Method. Spectral Element Methods benefit from the geometric flexibility of FEM and the convergence rates from spectral techniques. In the higher-order methods, spectral element methods and hp-version of FEM have been included as well as the p-version of FEM mentioned before. Higher-degree polynomials are preferred for their accuracy in smooth problems, thus this approach succeeds the geometrical flexibility in comparison with other spectral methods. The spectral FEM provides better accuracy for a fixed number of degrees of freedom. Finally, the stiffness matrix tends to be diagonal whereas in the *h-version* is full. [20]

2.5 Convergence and Error Estimates

In terms of convergence, it has been proved that in *p-* and *hp-versions* the rate is significantly better than the classical *h-version*, especially in smooth problems. More specifically in the *h-version*, convergence is limited by the degree of the polynomial where it is fixed. When the mesh is designed to be optimal the rate can be higher. The *h-version* of the finite element method can never achieve a convergence rate better than algebraic, independent of the mesh utilized, whereas the *p-*, *h-p version* has the potential to yield exponential convergence rates [8, 20].

For the three cases mentioned before the error estimates have the following forms. When a sequence of meshes is produced using uniform refinement, the estimate in the *h-version* is

$$||e||_E \leq C_1 N^{-\frac{1}{2}\min(p,\lambda)} \approx C_1 h^{\min(p,\lambda)}, \tag{8}$$

where C_1 is a positive constant, N is the nymber of degrees of freedom p is the polynomial degree, h is the element size and λ represents the 'strength' of the singularities. In the *p-version* where the mesh is fixed and the degree of elements changes the estimate is

$$||e||_E \leq C_2 N^{-\lambda}, \tag{9}$$

In a similar manner, for the *hp-version* the estimate is

$$||e||_E \leq C_3 e^{-\gamma N^{\theta}}, \tag{10}$$

where γ, θ are constants. In this case, the rate of convergence is exponential. [23, 26]

2.6 Adaptive

Sometimes, accuracy problems arise in particular areas of the grid or mesh. Rather than refining the entire region uniformly, it is more effective to refine only the regions that require additional precision. This is called Adaptive Mesh Refinement (AMR), which has a variety of applications in engineering. AMR can be divided into three main categories: *h-refinement*, *p-refinement*, and *r-refinement*. In *h-refinement*, the type of elements is the same in the domain but the number and size change based on the geometry. In *p-refinement*, higher-order polynomials are used as the basis for the shape functions, while the mesh element size is constant. In *r-refinement*, the main feature is that the nodes are relocated in areas that require optimization. The number of nodes and elements here has not been altered. The aforementioned methods can be used in conjunction with one another, such as *hp-refinement*, *hr-method* etc. [25, 26]

3 Applications in Fluid Mechanics

This section is dedicated to applications. In the following problems, the main focus is on the effects of the increase in the degree of the polynomials. More specifically, to demonstrate how the mesh changes as the degree of the polynomial increases, adaptive mesh refinement is used.

3.1 The Poisson Equation

Assume the following problem

$$\begin{cases} -\Delta u = f, & in \quad \Omega \\ \quad u = u_D, & on \quad \partial\Omega \end{cases} \tag{11}$$

where

$$u_D = \begin{cases} 1, & x = 0 \\ 0, & x = 1 \end{cases}$$

The weak form of the finite element method is:

$$\int_\Omega \nabla u \nabla v = \int_\Omega f v \tag{12}$$

$$a(u, v) = (f, v) \tag{13}$$

In this problem $f(x, y) = 10e^{-\frac{(x-0.5)^2+(y-0.5)^2}{0.02}}$.

The domain is a unit square with 128 elements in the initial mesh, as it is shown below:

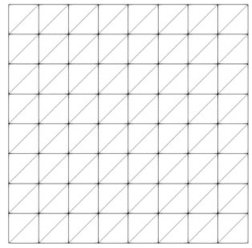
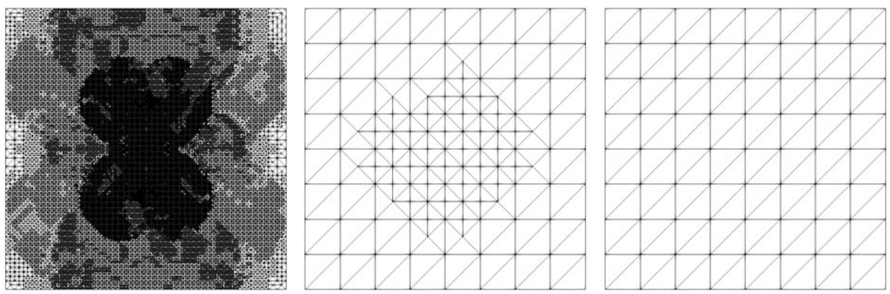


Fig. 2: Initial Mesh with 128 Elements



a) b) c)

Fig. 3: Adaptive Meshes with a) 57110 b) 234 and c) 128 Elements

In each case, the initial mesh has 128 elements. It can be easily seen that as the order of the polynomials is increased, the adapted mesh needs fewer elements for the solution without sacrificing the velocity results. In the third picture *c)* the adapted mesh is the same as the initial, the only difference is the increase of the degree from 1 to 3. The exact number of elements in meshes is 57110 when there are first-order polynomials, 234 for second-order, and 128 for third-order polynomials.

error	initial mesh	adapted mesh
k=1	1.72055	1.716084
k=2	1.71829	1.716344
k=3	1.71824	1.718240

Table 1: Error in L_2 norm.

The error in both the original case as well as in the adapted case decreases as the order increases.

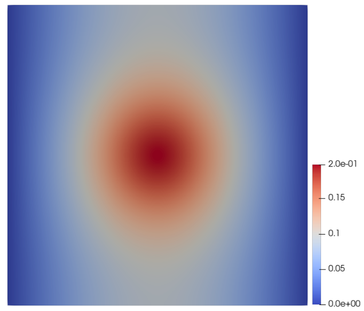


Fig. 4: Numerical Solution of the Poisson Equation

3.2 The Stokes Equation

The following problem is the Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad in \;\; \Omega \\ \nabla \cdot \mathbf{u} = 0 \quad in \;\; \Omega \\ \mathbf{u} = 0 \quad on \;\; \partial\Omega. \end{cases} \tag{14}$$

where ν denotes the kinematic viscosity and $\Omega \subset \mathbb{R}^2$ is a bounded domain. The function $\mathbf{u}$ denotes the velocity and p the pressure. The corresponding weak form of the Stokes equation is

$$\nu(\nabla\mathbf{u}, \nabla\mathbf{v}) - (\nabla \cdot \mathbf{v}, p) = (f, v) \tag{15}$$

$$(\nabla \cdot \mathbf{u}, q) = 0 \tag{16}$$

In the cases studied below a mixed function space $W = V \times Q$ is utilized with Taylor-Hood Elements. Taylor-Hood is a mixed element containing the (P_k, P_{k-1}) pair of polynomials with $k \geq 2$.

3.2.1 Backward Facing Step

A well-known test problem of the Stokes problem, for internal flows, is the Backward Facing Step. Due to the geometry, it creates a recirculation zone near the wall of the step.

The problem has the following formulation

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad in \ \ \Omega \\ \nabla\mathbf{u}\cdot\ \mathbf{n} + p\mathbf{n} = g \quad in \ \ \Gamma_N \\ \qquad\qquad \mathbf{u} = \mathbf{u}_D \quad on \ \ \Gamma_D. \end{cases} \tag{17}$$

where Γ_D, Γ_N are the Dirichlet and Neumann boundary conditions, respectively. The weak form is :

$$\int_\Omega [\nu\nabla\mathbf{u}\cdot\nabla\mathbf{v} - (\nabla\cdot\mathbf{v})p + (\nabla\cdot\mathbf{u})q]\, d\Omega = \int_\Omega \mathbf{f}\cdot\mathbf{v} d\Omega + \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v} ds$$
$$a\left((\mathbf{u}, p), (\mathbf{v}, q)\right) = L(\mathbf{v}, q)$$

Initially, the domain has 890 elements, and the adapted mesh with the (P_2, P_1) space has 1131 elements. As the polynomial order increases the number of elements decreases. Mesh tends to initial one when $k = 10$ and $((P_{10}, P_9))$ polynomials are used.

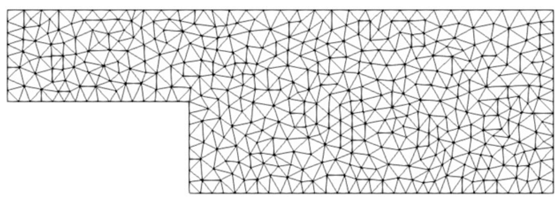


Fig. 5: Initial Mesh with 890 Elements

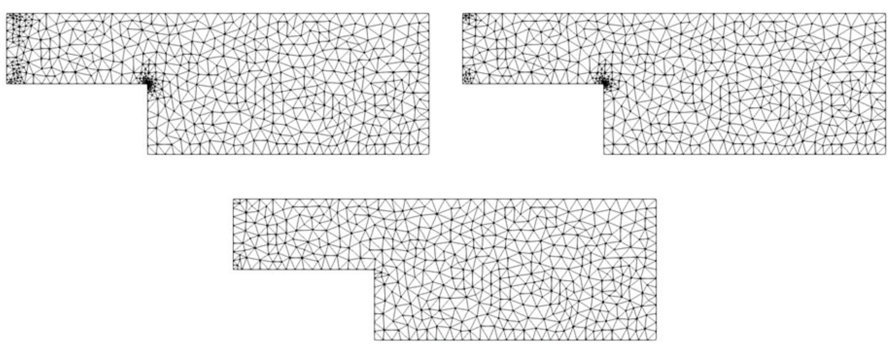


Fig. 6: Adaptive Meshes with

error	initial mesh	adapted mesh
(P_2, P_1)	0.912279	0.911745
(P_3, P_2)	0.911585	0.911762
(P_4, P_3)	0.911806	0.911780
(P_5, P_4)	0.911766	0.911765
(P_6, P_5)	0.911816	0.911774
(P_7, P_6)	0.911788	0.911769
(P_8, P_7)	0.911805	0.911775
(P_9, P_8)	0.911787	0.911778
(P_{10}, P_9)	0.911796	0.911796

Table 2: Error in L_2 norm.

The error in this problem is decreasing as the number of elements tends to the initial number of elements. Fluctuations in errors occur because in some cases there is difficulty in reducing the number of elements, resulting in either the number remaining constant or increasing.

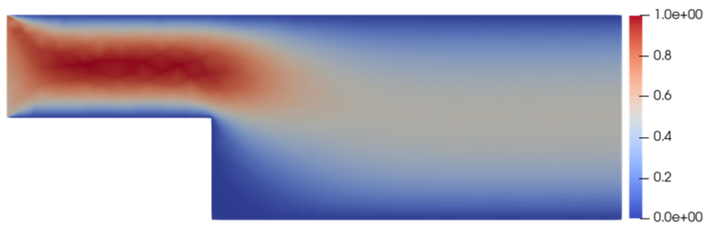


Fig. 7: Numerical Solution of the Backward Step

3.2.2 Lid Driven Cavity

The Lid Driven Cavity problem is another benchmark problem that is used in fluid mechanics. This problem is mainly studied due to the fact that exhibits a variety of phenomena that occur in incompressible flows such as secondary flows, complex flow patterns, turbulence, etc. There is a square domain with three rigid walls with no-slip conditions and a moving lid. In this problem, velocity is set to be equal to 1.

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad in \;\; \Omega \\ \nabla\mathbf{u} = 0 \quad in \;\; \Gamma_N \end{cases} \tag{18}$$

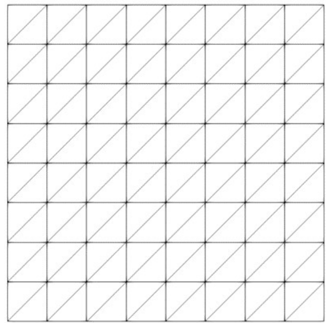


Fig. 8: Initial Mesh with 128 Elements.

In the beginning, the mesh has 128 and the adapted one 4669 elements, but as the order of the polynomials is increased the number of elements in the adapted mesh tends to become the same as the starting one.

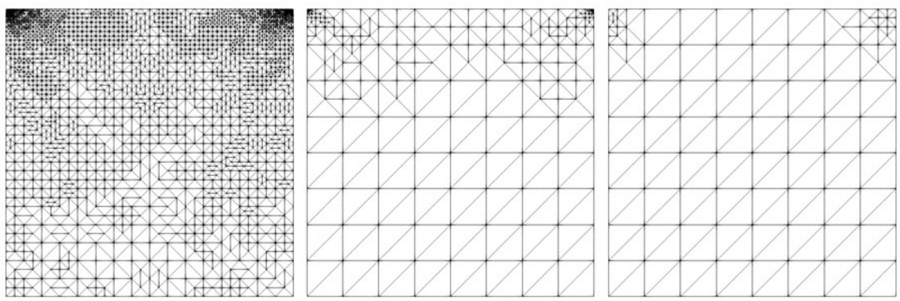


Fig. 9: Adaptive Meshes with a) 4669, b) 391 and c) 173 Elements

error	initial mesh	adapted mesh
(P_2, P_1)	0.75425947	0.75309220
(P_3, P_2)	0.75317886	0.75308080
(P_4, P_3)	0.75317088	0.75309430
(P_5, P_4)	0.75307177	0.75309132
(P_6, P_5)	0.75314475	0.75312226
(P_7, P_6)	0.75310587	0.75307011

Table 3: Error in L_2 norm.

As in the previous cases, a reduction of the error is observed as the degree of the polynomials increases. In some cases increasing the degree led to an increase in the mesh elements and this can be observed also in the errors table.

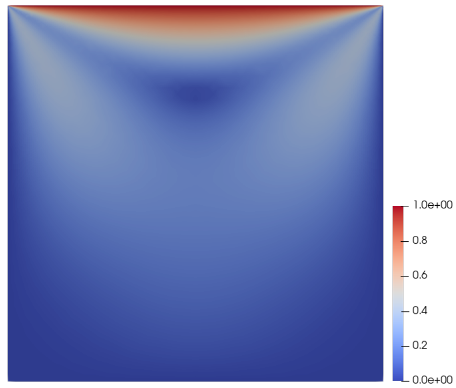


Fig. 10: Numerical Solution of the Lid Driven Cavity Problem

4 Conclusions

Utilizing high-order elements in the finite element method offers advantages, including providing more precise solutions than low-order elements, while simultaneously using fewer elements. This leads to a more efficient solution process.

In this review, the basic theory of the finite element has been presented by analysing the shape functions. The three alternate versions of FEM controlling the mesh and the polynomial degree were discussed, mainly focusing on higher-order polynomials and the *p-version* of FEM. In the applications section fluid mechanics problems were studied. It can be easily pointed out that the increase in the polynomial degree leads to the solution but with the minimum number of elements possible. This happens since higher-order polynomials can better approximate the functions.

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